is more convenient to speak of the variable x than the variable number represented by x. Hence the symbol definition of a variable has a slight advantage with respect to brevity and it makes it easier for those whose desire to avoid the term variable number.

The main object of the preceding remarks is to call attention to the principal elements involved in the different definitions of the term variable, and to give some reasons for the existence of these differences. The teacher is frequently called on to justify his views on points where different usages exist and it is desirable to be able to give references as well as reasons even if the difference appears somewhat trivial. If all numbers were essentially constants I presume most of us would agree with the foot-note quoted from Cesaro to the effect that there is no such thing as an infinite number, for we cannot conceive of any constant number which has such a distinctive character; but it is not so difficult to conceive of a variable which increases without limit and it is convenient to express this concept by saying that the variable is infinite. The ideas of infinite number and variable number are therefore closely related. The existence of the number zero, on the contrary, does not necessarily involve the concept of variability.

NOTE ON A FORMULA FOR THE SUMMATION OF CERTAIN POWER SERIES.

By HENRY KEMMERLING, Scranton, Pa.

Weierstrass's factor-theorem* is

$$fx = e^{Gx} \prod_{n=1}^{\infty} \left(1 - \frac{x}{a_n}\right) e^{x/a_n + \frac{1}{2}(x/a_n)^2 + \frac{1}{3}(x/a_n)^3 + \cdots + [1/(n-1)](x/a_n)^{n-1}},$$

where fx has zeros at the points $a_1, a_2, a_3, \ldots$

Taking logarithms and differentiating,

$$\frac{f'x}{fx} = G'x + \sum_{n=1}^{\infty} \left(\frac{1}{x-a_n} + \frac{1}{a_n} + \frac{x}{a_n^2} + \cdots + \frac{x^{n-2}}{a_n^{n-1}}\right).$$

Then if $\left(\frac{f'x}{fx} - G'x\right)$ is Px, we have

*Harkness and Morley's *Introduction to Analytic Functions*, p. 199

$$Px=\sum_{n=1}^{\infty}\left(\frac{1}{x-a_n}+\frac{1}{a_n}+\frac{x}{a_n^{2}}=\ldots.\frac{x^{n-2}}{a_n^{n-1}}\right)=-\sum_{n=1}^{\infty}\left(\frac{x^{n-1}}{a_n^{n}}+\frac{x^{n}}{a_n^{n+1}}+\ldots.\right). \tag{1}$$

Maclauren's formula is

$$Px=P0+P'0\frac{x}{1}+\ldots. P^{n}0\frac{x^{n}}{n!}+\ldots. \tag{2}$$

Comparing the general terms of (1) and (2), we have

$$P^{n}0\frac{x^{n}}{n!}=-\sum_{n=1}^{\infty}\frac{x^{n}}{a_n^{n-1}},\text{ or }P^{n}0=-n!\sum_{n=1}^{\infty}\frac{1}{a_n^{n+1}}. \tag{3}$$

As an application of (3) take $fx=x^3-6x^2+11x-6$, whose zeros a_1, a_2, a_3, are 1, 2, 3.

$f'x=3x^2-12x+11$.

$$Px=\frac{f'x}{fx}=\frac{11-12x+3x^2}{-6+11x-6x^2+x^3},$$

which by division,

$$=-\tfrac{11}{6}-\tfrac{49}{36}x-\tfrac{251}{216}x^2-\ldots.$$

Using (3), and reducing, we have,

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}=\tfrac{11}{6},$$

$$\frac{1}{a_1^{2}}+\frac{1}{a_2^{2}}+\frac{1}{a_3^{2}}=\tfrac{49}{36},$$

$$\frac{1}{a_1^{3}}+\frac{1}{a_2^{3}}+\frac{1}{a_3^{3}}=\tfrac{251}{216},$$

. . . . . .

Again let $fx=1-\sin\frac{\pi x}{2}$, which has *two* zeros at each of the points $+1, -3, +5, -7, \ldots.$

$$f'x=-\frac{\pi}{2}\cos\frac{\pi x}{2},$$

$$Px=\frac{f'x}{fx}=\frac{-(\pi/2)\cos(\pi x/2)}{1-\sin(\pi x/2)},$$

$$P'x=\frac{-(\pi^2/4)}{1-\sin(\pi x/2)},$$

$$P''x=\frac{-(\pi^3/8)\cos(\pi x/2)}{[1-\sin(\pi x/2)]^2},$$

$$P'''x=\frac{(\pi^4/16)\sin(\pi x/2)[1-\sin(\pi x/2)]-(\pi^4/8)\cos^2(\pi x/2)}{[1-\sin(\pi x/2)]^3}.$$

$P0=-\pi/2$, $P'0=-\pi^2/4$, $P''0=-\pi^3/8$, $P'''0=-\pi^4/8$.

Applying (3),

$$-\frac{\pi}{2}=-2\left(\frac{1}{1}+\frac{1}{-3}+\frac{1}{5}+\frac{1}{-7}+\ldots.\right)$$

$$-\frac{\pi^2}{4}=-2\left(\frac{1}{1^2}+\frac{1}{(-3)^2}+\frac{1}{5^2}+\frac{1}{(-7)^2}+\ldots.\right)$$

$$-\frac{\pi^3}{8}=-4\left(\frac{1}{1^3}+\frac{1}{(-3)^3}+\frac{1}{5^3}+\frac{1}{(-7)^3}+\ldots.\right)$$

$$-\frac{\pi^4}{8}=-12\left(\frac{1}{1^4}+\frac{1}{(-3)^4}+\frac{1}{5^4}+\frac{1}{(-7)^4}+\ldots.\right)$$

Reducing above we get the familiar series,

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots.$$

$$\frac{\pi^3}{32}=\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots.$$

$$\frac{\pi^4}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ldots.$$

. . . . . .

NOTES ON THE RADICAL AXIS AND THE QUADRILATERAL.

By WILLIAM GALLATLY, Boscombe, England.

I. THE RADICAL AXIS.

1. If d be the distance of a point P from the center of a circle X of radius x, then $\pm(d^2-x^2)$ is called the power of P with respect to X. ABC being the triangle of reference, let ρ_1^2, ρ_2^2, ρ_3^2 be the powers of A, B, C, and let the circle X have the equation

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+k(a\alpha+b\beta+c\gamma)(l\alpha+m\beta+n\gamma)=0.$$

Now if $S=0$ be any conic, and if S' be the value of S when the coordinates of P, Cartesian or trilinear, are substituted for the general coordinates, it is known that S' is proportional to the power of P. Hence, for the point A, $\lambda\rho_1^2=k.2\triangle.l.2\triangle/a$; for A is $(2\triangle/a, 0, 0)$. Hence $kl=\lambda/4\triangle^2.a\rho_1^2$.

The radical axis of the circles X and ABC is thus

$$a\rho_1^2.\alpha+b\rho_2^2.\beta+c\rho_3^2.\gamma=0.$$

And the circle X takes the form

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+\frac{\lambda}{4\triangle^2}.(a\alpha+b\beta+c\gamma)(a\rho_1^2.\alpha+b\rho_2^2.\beta+c\rho_3^2.\gamma)=0.$$

2. We now proceed to find the radical axis of the circle ABC, and (1) the in-circle; (2) the nine point circle; (3) the polar circle; (4) the circle on Hb (H being the orthocenter).

For (1), $\rho_1^2=(s-a)^2$; so that the radical axis is $a(s-a)^2.\alpha+\ldots.=0$.

For (2), N being the nine point center,

$$\rho_1^2=AN^2-Nd^2=Ad^2+2.Ad.dN.\cos(B-C)$$
$$=R^2\cos A[\cos A+\cos(B-C)]=2R^2\cos A\ \sin B\ \sin C.$$

Hence $a\rho_1^2$ varies as $\cos A$. The radical axis is $\cos A.\alpha+\ldots.=0$.

For (3). The polar circle has center H, and $x^2=-4R^2\cos A\ \cos B\ \cos C$. Therefore, $\rho_1^2=AH^2-k^2=4R^2\cos^2 A+4R^2\cos A\ \cos B\ \cos C=4R^2\ \cos A\ \sin B \times\sin C$. Hence $a\rho_1^2$ varies as $\cos A$, and the radical axis is $\cos A.\alpha+\ldots.=0$.

For (4). Let ω be the mid-point of GH.

$$2\rho_1^2=2(A\omega^2-\omega H^2)=2(A\omega^2+\omega H^2)-Hb^2=Ab^2+AH^2-HG^2$$
$$=2AG.AH.\cos GAH=2AH.Ag=2.AH.\tfrac{3}{2}.AD'$$
$$=4R\ \cos A.\tfrac{3}{2}.2R\ \sin B\ \sin C.$$

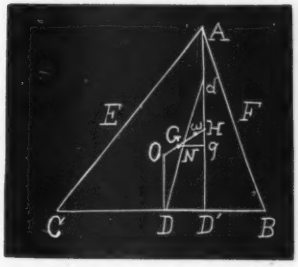
Hence $a\rho_1^2$ varies as $\cos A$, and the radical axis is $\cos A.\alpha+....=0$.

3. To determine the radical axis of the nine point circle, and (1) circum-circle, (2) in-circle.

It is now convenient to take the mid-point triangle *DEF as triangle of reference.*

If $(\alpha'\beta'\gamma')$ and $(\alpha\beta\gamma)$ be the old and new coordinates of a point, then $\alpha'+\alpha=\frac{1}{2}.AD'=2\triangle/a$; all symbols now referring to *DEF*, so that $a\alpha'=b\beta+c\gamma$; and $l\alpha+m\beta+n\gamma=0$ becomes

$$a\alpha\left(\frac{m}{b}+\frac{n}{c}\right)+....=0.$$



Hence $\cos A.\alpha+....=0$, becomes $a^3\alpha+....=0$.

For (1), $x=2R$; $d=OD=2R\cos A$.

Hence $\rho_1^2=4R^2(1-\cos^2 A)$. Therefore $a\rho_1^2$ varies as a^3, so that the radical axis is $a^3.\alpha+....=0$, agreeing with Art. 2 (1).

For (2), $\rho_1^2=(b-c)^2$; so that the radical axis is $a(b-c)^2.\alpha+b(c-a)^2.\beta+c(a-b)^2.\gamma=0$. But this is the tangent to $a/\alpha+....=0$ (the nine point circle) at

$$\left(\frac{1}{b-c},\ \frac{1}{c-a},\ \frac{1}{a-b}\right).$$

Hence (Feuerbach's Theorem), the nine point and in-circles touch; the point of contact being

$$\left(\frac{1}{b-c},\ \frac{1}{c-a},\ \frac{1}{a-b}\right).$$

II. The Quadrilateral.

1. Let the triangle ABC be cut by the straight line

$$\frac{a\alpha}{l}+\frac{b\beta}{m}+\frac{c\gamma}{n}=0$$

in the points A', B', C'. Bisect AA', BB', CC' in A_1, B_1, C_1.

For A_1, $\frac{b\beta}{m}+\frac{c\gamma}{n}=0$, and $b\beta+c\gamma=a\alpha$.

$\therefore \frac{b\beta}{m}=\frac{c\gamma}{-n}=\frac{a\alpha}{m-n}=\frac{\triangle}{m-n'}$, with similar expressions for B'. Hence

$$A'B' \text{ is } (m+n-l)a\alpha+(n+l-m)b\beta+(l+m-n)c\gamma=0,$$

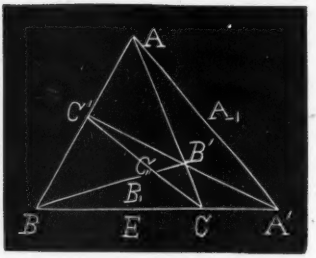
the symmetry of the equation showing that C' is also on this line.

2. If $BCB'C'$ is circumscriptible to a circle, then the mid-point line passes through the center. The in-circle of ABC is $\surd[a(s-a).\alpha+....=0$.

If $A'B'C'$ is a tangent, then $\frac{a(s-a)}{a/l}+....=0$, or $(m+n-l)a+....=0$.

Hence the centre (1, 1, 1) lies on the mid-point line.

3. The orthocenters of the four triangles ABC, $AB'C'$, $BC'A'$, $CA'B'$, are collinear.



The perpendiculars from B' on AB, and C' on AC, are found to be

$$na.\cos A.\alpha+(lc-na\cos B)\beta+lc\cos A.\gamma=0.$$
$$ma.\cos A.\alpha+lb\cos A.\beta+(lb-ma\cos c)\gamma=0.$$

By subtraction we obtain the symmetrical equation

$$(m-n)\cos A.\alpha+(n-l)\cos B.\beta+(l-m)\cos C.\gamma=0,$$

which proves the proposition.

4. To find the equations of the circles $AB'C'$, $BC'A'$, $CA'B'$. Let the equation of $AB'C'$ be

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=k(a\alpha+b\beta+c\gamma)(m'\beta+n'\gamma)\quad (l' \text{ obviously zero}).$$

At B', $\frac{a\alpha}{l}+\frac{c\gamma}{n}=0$; $\beta=0$. Hence, $\frac{a\alpha}{-l}=\frac{c\gamma}{n}=\frac{2\triangle}{n-l}$; and $b\gamma\alpha=k.2\triangle.n'\gamma$.

Hence, $kn'=\frac{b}{a}.\frac{-l}{n-l}$; so $km'=\frac{c}{a}.\frac{l}{l-m}$.

Hence, the equation of $AB'C'$ is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=\frac{l}{a}(a\alpha+b\beta+c\gamma)\left(\frac{c}{l-m}.\beta-\frac{b}{n-l}.\gamma\right);$$

so that, if the circle $AB'C'$ cuts the circle ABC in D, the equation to AD is

$$\frac{c}{l-m}.\beta-\frac{b}{n-l}.\gamma=0, \text{ or } \beta\div\frac{b}{n-l}=\gamma\div\frac{c}{l-m}.$$

It follows that the circles $AB'C'$, $BC'A'$, $CA'B$ cut the circle ABC in the same point D, whose coordinates are $\left(\frac{a}{m-n}, \frac{b}{n-l}, \frac{c}{l-m}\right)$.

5. The condition that $BCB'C'$ may be cyclic, or that $B'C'$ may be anti-parallel to BC. In this case it can be at once proved geometrically that D lies on AA' or $\frac{b\beta}{m}+\frac{c\gamma}{n}=0$. Hence the required condition is

$$\frac{b^2}{m(m-n)}+\frac{c^2}{n(l-m)}=0.$$

6. The Simson Line of D.

The Simson line of $(\alpha'\beta'\gamma')$ is known to be

$$\frac{a^2}{\alpha'^2}\cdot\frac{a}{\beta'\cos C-\gamma'\cos B}+\ldots.=0.$$

So that the Simson line of D is

$$\frac{m-n}{la-mb\cos C-nc\cos B}\cdot a+\ldots.=0.$$

The equation to Simson line which makes angles θ_1, θ_2, θ_3 with the sides of ABC is $a\cot\theta_1.\alpha+\ldots.=0$.

Hence, for the Simson line of D,

$$a\cot\theta_1 \text{ varies as } \frac{m-n}{la-mb\cos C-nc\cos B}.$$

7. To determine the radical axis of the circle on AA' as diameter, and the circle ABC.

Applying the 'power' method to the circle AA',

$$\begin{aligned}2\rho_2^2&=2(BA_1^2-AA_1^2)=2(BA_1^2+AA_1^2)-AA'^2\\&=AB^2+A'B^2-AA'^2=2.AB.A'B.\cos B.\end{aligned}$$

Hence, $b\rho_2^2=abc\cos B.\frac{n}{n-m}$. So $c\rho_3^2=-abc\cos C.\frac{m}{n-m}$. So that the radical axis is $n\cos B.\beta-m\cos C\gamma=0$. And the circle on AA' is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+\frac{\lambda.abc}{4\triangle^2}(a\alpha+b\beta+c\gamma)\left(\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}\right)=0.$$

Draw AE perpendicular to BC, then E lies on the circle AA'. Its co-ordinates are $(0,\ b\sin C\cos C,\ c\sin B\cos B)$. Hence the circle AA' is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta-(a\alpha+b\beta+c\gamma)\left(\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}\right)=0.$$

It follows that the radical axis of AA', BB', is

$$\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}-\frac{n\cos A.\alpha-l\cos C.\gamma}{n-l}=0,$$

$$\text{or } (m-n)\cos A.\alpha+(n-l)\cos B.\beta+(l-m)\cos C.\gamma=0,$$

the symmetry of the equation showing that it is the radical axis of all three circles AA', BB', CC'. This radical axis is also the line of the four orthocenters (Art. 3).

8. The radical axis of the diagonal circles is parallel to the Simson Line of D.

If $\lambda\alpha+\mu\beta+\nu\gamma=0$ makes an angle θ with BC, then

$$\cot\theta=\frac{\lambda-\mu\cos C-\nu\cos B}{\nu\sin B-\mu\sin C},$$

so that for the radical axis,

$$a\cot\theta \text{ varies as } \frac{(m-n)}{la-mb\cos C-nc\cos B}.$$

Hence, the radical axis is parallel to the Simson Line (Art. 3).

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Note on Problems 267 and 268, by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

In the *Messenger of Mathematics*, 1874, Vol. III, page 137, Mr. Glaisher drew attention to the formula:

$$\tan nx=\frac{n\sin x}{\cos x+}\frac{(1^2-n^2)\sin^2 x}{3\cos x+}\frac{(2^2-n^2)\sin^2 x}{5\cos x+}\frac{(3^2-n^2)\sin^2 x}{7\cos x+\ldots.} \qquad (1)$$

x being $<\frac{1}{2}\pi$, and n unrestricted.

For $x=\frac{1}{4}\pi$, $\tan\frac{n\pi}{4}=\cfrac{n}{1+}\cfrac{1^2-n^2}{3+}\cfrac{2^2-n^2}{5+}\cfrac{3^2-n^2}{7+\ldots.}$

Put $n=\frac{1}{2}$, $\tan\frac{\pi}{8}=\cfrac{1}{2+}\cfrac{1}{2+}\cfrac{1}{2+\ldots}$

Put $n=\frac{1}{3}$, $\tan\frac{\pi}{12}=\cfrac{3}{9+}\cfrac{6.12}{27+}\cfrac{15.21}{45+}\cfrac{24.30}{63+\ldots}$

Put $n=ni$, $\tanh\frac{n\pi}{4}=\cfrac{n}{1+}\cfrac{n^2+1}{3+\ldots.}$

and as $e^{2x}=-1+\cfrac{2}{1-\tanh x}$, this gives a continued fraction for $e^{\ln x}$.

On page 65 of Vol. IV, 1875, Mr. Glaisher states that the value for $\tan nx$ was not, as he had thought, original. Vorsselmann and Heer of Utrecht quoted it in 1833 from Euler (*Mém. de l'Acad. de Pétersbourg*, 1813) in the form

$$\tan nx=\cfrac{n\tan x}{1-}\cfrac{(n^2-1)\tan^2 x}{3-}\cfrac{(n^2-2^2)\tan^2 x}{5-}\cfrac{(n^2-3^2)\tan^2 x}{7-\ldots.} \qquad (2)$$

Euler had derived it thus:

$$nz\,\frac{(1+z)^n+(1-z)^n}{(1+z)^n-(1-z)^n}=1+\cfrac{(n^2-1)z^2}{3+}\cfrac{(n^2-2^2)z^2}{3+}\cfrac{(n^2-3^2)z^2}{5+\ldots.} \qquad (3)$$

Vorsellmann derived it from a transformation of

$$\frac{F(\beta+\gamma,\ \beta+1,\ \gamma+1,\ x)}{F(\beta+\gamma,\ \beta,\ \gamma,\ x)}.$$

Glaisher got it from the differential equation corresponding to $y=\cos(n\cos^{-1}x)$ *i. e.*, from $(1-x^2)y_2-xy_1+n^2y=0$ differentiated m times, *i. e.*, $(1-x^2)y_{m+2}-(2m+1)y_{m+1}x+(n^2-m^2)y_m=0$. Then replacing x by $\cos x$, (1) follows at once. Vorsellmann also gave as his own:

$$\tan nx=\cfrac{nt}{1-t^2-}\cfrac{(n^2-4)t^2}{3(1-t^2)-}\cfrac{(n^2-16)t^2}{5(1-t^2)-\ldots} \qquad (4),\ \text{terminating when } n \text{ is even,}$$

$$=\cfrac{nt}{1-}\cfrac{(n^2-1)t^2}{3-t^2-}\cfrac{(n^2-9)t^2}{5-3t^2-\ldots} \qquad (5),\ \text{terminating when } n \text{ is odd, } 2t=\tan x.$$

Glaisher points out that (4) is gotten from (2) by substituting $2x$ for x and $\frac{2\tan x}{1-\tan^2 x}$ for $\tan 2x$, and $\frac{1}{2}n$ for n.

In the same way Glaisher gets

$$\tan nx = \frac{nt(1-t^2)}{1-6t^2+t^4-} \frac{(n^2-16)t^2(1-t^2)^2}{3-18t^2+t^4-} \frac{(n^2-64)t^2(1-t^2)^2}{5-30t^2+5t^4-\ldots.}$$

terminating when n is a multiple of 4.

Putting $n=0$ in (2), (4), (5), we get

$$\tan^{-1}x = \frac{x}{1+} \frac{x^2}{3+} \frac{4x^2}{5+\ldots}, \quad (6) \text{ well known.}$$

$$\tan^{-1}x = \frac{x}{1-x^2+} \frac{4x^2}{3(1-x^2)+} \frac{16x^2}{5(1-x^2)+\ldots.} \quad (7)$$

which is (6) with $\frac{2x}{1-x^2}$ for x.

$$\tan^{-1}x = \frac{x}{1+} \frac{x^2}{3-x^2+} \frac{9x^2}{5-x^2+\ldots.}, \quad (8) \text{ due to Euler (1779).}$$

Since $\tan nx$ only changes sign if ni be substituted for n and xi for x, (2) gives us

$$\tan nx = \frac{n\tanh x}{1-} \frac{(n^2+1)\tanh^2 x}{3-} \frac{(n^2 4)+\tanh^2 x}{5-\ldots}$$

$$\text{or } \tan\left[n \log\sqrt{\frac{1+x}{1-x}}\right] = \frac{nx}{1-} \frac{(n^2+1)x^2}{3-} \frac{(n^2+4)x^2}{5-\ldots.},$$

a formula also obtainable by replacing n by ni in (3).

Hence, by giving special values to x in (2), (4), and (5), and replacing n by x, we get

$$\tan\frac{\pi x}{4} = \frac{x}{1-} \frac{x^2-1}{3-} \frac{x^2-4}{5-\ldots} = \frac{x}{1-} \frac{x^2-1}{2-} \frac{x^2-9}{2-\ldots}$$

$$\tan\frac{\pi x}{3\sqrt{3}} = \frac{x}{1-} \frac{x^2-3}{3-} \frac{x^2-12}{5-\ldots} = \frac{2x}{3-} \frac{4x^2-3}{9-} \frac{4x^2-12}{15-\ldots.} = \frac{2x}{3-} \frac{4x^2-3}{8-} \frac{4x^2-27}{12-\ldots.}$$

$$\tan\frac{\pi x}{6}=\frac{x}{\sqrt{3}-}\,\frac{x^2-1}{3\sqrt{3}-}\,\frac{x^2-4}{5\sqrt{5}-}$$

For $x=0$, $\pi=\dfrac{3\sqrt{3}}{1+}\,\dfrac{3.1^2}{3+}\,\dfrac{3.2^2}{5+}\,\dfrac{3.3^2}{7+\ldots}=\dfrac{6\sqrt{3}}{3+}\,\dfrac{3.1^2}{9+}\,\dfrac{3.2^2}{15+}\,\dfrac{3.3^2}{21+\ldots}$

$$=\frac{6\sqrt{3}}{3+}\,\frac{3.1^2}{8+}\,\frac{3.3^2}{12+}\,\frac{3.5^2}{16+\ldots}=\frac{6}{\sqrt{3}+}\,\frac{1}{3\sqrt{3}+}\,\frac{2^2}{5\sqrt{3}+}\,\frac{3^2}{7\sqrt{3}+\ldots.}$$

which may also be deduced from (6), (7), and (8).

Glaisher points out the specific advantage of getting a continued fraction from a differential equation, for not only is the result obtained *ab initio* but the remainder at any stage is exhibited in a finite form. He takes $y=\cos(\sin^{-1}\sqrt{z})$ and its differential equation $4(z-z^2)y_2+2(1-2z)y_1+n^2y=0$. Differentiate m times and we have

$$4(z-z^2)y_{m+2}+2(2m+1)(1-2z)y_{m+1}+(n^2-4m^2)y_m=0.$$

Put $z^2=\sin^2\frac{1}{2}x$ and $2n$ for n and we deduce (4).

From $y=\cos\sqrt{x}$ we get $\tan x=\dfrac{x}{1-}\,\dfrac{x^2}{3-\ldots}$

In Vol. VII (1878), page 67, the same author gives further interesting forms:

$$\frac{\log[x+\sqrt{(1+x^2)}]}{\sqrt{(1+x^2)}}=x-\tfrac{2}{3}x^3+\frac{2.4}{3.5}x^5-\frac{2.4.6}{3.5.7}x^7+\ldots.$$

$$=\frac{x}{1+}\,\frac{1.2x^2}{3-2x^2+}\,\frac{3.4x^2}{5-4x^2+}\,\frac{5.6x^2}{7-6x^2+\ldots}=\frac{1}{x^{-1}+}\,\frac{1.2}{3x^{-1}-2x+\ldots}$$

$$\frac{\sin^{-1}x}{\sqrt{(1-x^2)}}=x+\tfrac{2}{3}x^3+\frac{2.4}{3.5}x^5+\frac{2.4.6}{3.5.7}x^7+\ldots.$$

$$=\frac{x}{1-}\,\frac{1.2x^2}{2x^2+3-}\,\frac{3.4x^2}{4x^2+5-}\,\frac{5.6x^2}{6x^2+7-\ldots}=\frac{1}{x^{-1}-}\,\frac{1.2}{2x+3x^{-1}-\ldots.}$$

Here, putting $x/\sqrt{2}$, we get

$$\frac{2}{\sqrt{3}}\log\frac{1+\sqrt{3}}{\sqrt{2}}=\frac{1}{1+}\,\frac{1}{2+}\,\frac{6}{3+}\,\frac{15}{4+}\,\frac{28}{5+\ldots}$$

1, 6, 15, 28 being the alternate triangular numbers.

$$\frac{\sqrt{2}}{\sqrt{3}}\log\frac{1+\sqrt{3}}{\sqrt{2}}=\frac{1}{\sqrt{2}+}\ \frac{1.2}{2\sqrt{2}+}\ \frac{3.4}{3\sqrt{2}+..}$$

$$\frac{\pi}{2}=\frac{1}{1-}\ \frac{1}{4-}\ \frac{6}{7-}\ \frac{15}{10-}\ \frac{28}{13-...}$$

$$\frac{\pi}{2\sqrt{2}}=\frac{1}{\sqrt{2}-}\ \frac{1.2}{4\sqrt{2}-}\ \frac{3.4}{7\sqrt{2}-}\ \frac{5.6}{10\sqrt{2}-..}$$

For $x=\frac{1}{2}$, $\frac{2}{\sqrt{5}}\log\frac{1+\sqrt{5}}{2}=\frac{1}{2+}\ \frac{1.2}{5+}\ \frac{3.4}{8+}\ \frac{5.6}{11+...}$

$$\frac{\pi}{3\sqrt{3}}=\frac{1}{2-}\ \frac{1.2}{7-}\ \frac{3.4}{12-}\ \frac{5.6}{17-...}$$

For $x=1$ in the expansion of $\frac{\log[x+\sqrt{(1+x^2)}]}{\sqrt{(1+x^2)}}$

$\frac{\log(1+\sqrt{2})}{\sqrt{2}}=\frac{1}{1+}\ \frac{1.2}{1+}\ \frac{3.4}{1+}\ \frac{5.6}{1+...}$, and he compares this with

$$\frac{\pi}{2}-1=\frac{1}{1+}\ \frac{1.2}{1+}\ \frac{2.3}{1+}\ \frac{3.4}{1+}\ \frac{5.6}{1+...}$$

GEOMETRY.

318. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Is it possible by a straight edge and sect carrier, *i. e.*, without the use of a circle, to construct a mean proportional to two given sects?

Remark by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The value of the length of the mean proportional can be approximately measured without the application of the circle, but it cannot be constructed by pure geometry without such application.

318. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Given three radii and the distances apart of the centers of three circles, to find the radii of the eight circles touching the three given circles.

II. Solution by G. W. GREENWOOD, Dunbar, Pa.

Consider first the problem of describing a circle touching two given circles and passing through a given point. Invert with respect to the point; the circles in general invert into circles; draw any common tangent to them

and re-invert into the original circles. The common tangent inverts into a circle through the point and tangent to the given circles. There are, in general, four solutions.

Next, let A, B, C be the centers, and a, b, c the radii, of three given circles. Suppose a is not greater than b or c. Describe circles with radii $b-a$ and $c-a$, and centers B, C, respectively. Let X be the center and x the radius of a circle through A and tangent externally to these two circles. Then a circle, center X and radius $x-a$ will be tangent to the three given circles. By obvious modifications we can, in general, get seven more circles touching the given circles.

We can get the numerical values of the radii and the positions of the centers by means of the linear relations connecting inverse figures.

It would be interesting to consider all the possible cases arising in making these constructions. For example, in the case of the escribed circles of a triangle the sides of the triangle are the limits of three pairs of circles and the nine-point circle is one of the remaining two circles.

CALCULUS.

245. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

Prove or disprove:

$$\int_0^\infty \frac{dx}{\sqrt{(x^2-1)(k^2x^2-1)}} = 2F'(k) + \sqrt{(-1)}.F'[\sqrt{(1-k^2)}],$$

Legendre's notation, $0<k<1$.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$L=\int_0^\infty \frac{dx}{\sqrt{[(x^2-1)(k^2x^2-1)]}} = \int_0^1 \frac{dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}}$$

$$+\int_1^\infty \frac{dx}{\sqrt{[(x^2-1)(k^2x^2-1)]}} = F'(k) + \int_1^\infty \frac{dx}{\sqrt{[(x^2-1)(k^2x^2-1)]}}.$$

Let $x=\frac{1}{ky}$. Then $\int_0^\infty \frac{dx}{\sqrt{[(x^2-1)(k^2x^2-1)]}}$

$$= \int_0^{1/k} \frac{dy}{\sqrt{[(1-y^2)(1-k^2y^2)]}} = F'(k) + \int_1^{1/k} \frac{dy}{\sqrt{[(1-y^2)(1-k^2y^2)]}}.$$

$$\therefore L = 2F'(k) + \int_1^{1/k} \frac{dy}{\sqrt{[(1-y^2)(1-k^2y^2)]}}.$$

Let $y=\frac{1}{\sqrt{(1-k'^2z^2)}}$, where $k'=\sqrt{(1-k^2)}$.

$$\therefore dy=\frac{k'^2 z dz}{(1-k'^2 z^2)^{\frac{1}{2}}},\quad \frac{1}{\sqrt{[(1-y^2)(1-k^2y^2)]}}=\frac{\sqrt{(-1)}(1-k'^2z^2)}{k'^2z\sqrt{(1-z^2)}}.$$

$$\therefore \int_1^{1/k}\frac{dy}{\sqrt{[(1-y^2)(1-k^2y^2)]}}=\sqrt{(-1)}\int_0^1\frac{dz}{\sqrt{[(1-z^2)(1-k'^2z^2)]}}$$

$$=\sqrt{(-1)}F'(k').$$

$$\therefore L=2F'(k)+\sqrt{(-1)}F'[\sqrt{(1-k^2)}].$$

Also solved by the Proposer.

246. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

Derive Taylor's Series by the use of the formula for successive integration by parts, and nothing else.

Solution by the PROPOSER.

Assume the identity, $F(a+h)-F(a)=\int_a^{a+h}F'(x)dx$, which on integrating the right hand member by parts,

$$=-\left[(a+h-x)F'(x)\right]_a^{a+h}+\int_a^{a+h}(a+h-x)F''(x)dx$$

$$=hF'(a)-\left[\frac{1}{2!}(a+h-x)^2F''(x)\right]_a^{a+h}+\frac{1}{2!}\int_a^{a+h}(a+h-x)^2F'''(x)dx$$

$$=hF'(a)+\frac{h^2}{2!}F''(a)+....+\frac{h^n}{n!}F^n(a)+\frac{1}{n!}\int_a^{a+h}(a+h-x)^nF^{n+1}(x)dx.$$

Hence, $F(a+h)-\left[F(a)+hF'(a)+\frac{h^2}{2!}F''(a)+...+\frac{h^n}{n!}F^n(a)\right]$

$$(=R)=\frac{1}{n!}\int_a^{a+h}(a+h-x)^nF^{n+1}(x)dx.$$

We have assumed here that $F(x)$, $F'(x)$, $F''(x)$, ..., $F^n(x)$, $F^{n+1}(x)$, are all finite and continuous between the limits a and $(a+h)$ of x; now, if we put $A=F^{n+1}(a+\theta h)$, a mean value of $F^{n+1}(x)$ between the limits a and $(a+h)$, we have

$$R=\frac{1}{n!}\int_a^{a+h}(a+h-x)^nAdx=\frac{h^{n+1}}{(n+1)!}F^{n+1}(a+\theta h).$$

Also solved by Francis Rust.

MECHANICS.

205. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given two points A and B not in the same horizontal nor in the same vertical line; to find the path from A to B along which a particle will slide from rest under the force of gravity alone so that the average velocity along the curve shall be a maximum.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let the particle start from rest at the point (x_1, y_1), where the axis of x is vertical and the axis of y horizontal. Then

$$v=\frac{ds}{dt}=\sqrt{[2g(x-x_1)]}=\frac{dx}{dt}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Since v=a maximum, t=a minimum.

$$\therefore t=U=\int_{x_1}^{x_2}\frac{\sqrt{[1+(dy/dx)^2]}\,dx}{\sqrt{[2g(x-x_1)]}}=\text{minimum}.$$

By Calculus of Variations, since the expression in the right hand member does not contain y explicitly, the differential of this expression is equal to a constant.

$$\therefore \frac{dy/dx}{\sqrt{[2g(x-x_1)]}\,\sqrt{[1+(dy/dx)^2]}}=C.$$

$$\therefore \left(\frac{dy}{dx}\right)^2=C^2\left[1+\left(\frac{dy}{dx}\right)^2\right][2g(x-x_1)].$$

$$\therefore \frac{dy}{dx}=\pm\frac{C\sqrt{[2g(x-x_1)]}}{\sqrt{[1-2g\,C^2(x-x_1)]}},$$

the differential equation of a cycloid. This is Bernoulli's famous problem and is solved in almost every work on the Calculus of Variations as well as Dynamics.

206. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A rigid square $ABDC$ made by smooth wires is fixed with A vertically above D. Two small equal spherical elastic beads slide down BD, CD, starting simultaneously from B and C. Find the ratio of their velocities of approach and separation at D, and how far they will separate after impact.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The particles are sliding down inclined planes, inclined at an angle $\beta=\frac{1}{4}\pi$ to the horizon, starting from rest. Let a=side of square, x=distance each particle has moved after a time t. Then the velocity of each is

$$v=\sqrt{(2g\sin\tfrac{1}{4}\pi.x)}=\sqrt{(gx\sqrt{2})}=\sqrt{(ga\sqrt{2})} \text{ at } D.$$

Each particle is moving toward the diagonal at a velocity $=v\sin\frac{1}{4}\pi=\frac{1}{2}v\sqrt{2}=\frac{1}{2}\sqrt{[2\sqrt{(2)}gx]}=\frac{1}{2}\sqrt{[2\sqrt{(2)}ga]}$ at D. Hence, the particles at D approach each other with a velocity $=2\times\frac{1}{2}\sqrt{[2\sqrt{(2)}ga]}=\sqrt{[2\sqrt{(2)}ga]}$.

Let e=the coefficient of restitution. Then, since the particles impinge at right angles, they will separate with a velocity $v_1=e\sqrt{[2\sqrt{(2)}ga]}$, and will ascend DB, DA, respectively, with a velocity $v_2=e\sqrt{(ga\sqrt{2})}$. Each will ascend a distance $=v_2^2/2g\sin\frac{1}{4}\pi=v_2^2/g\sqrt{2}=e^2a$.

Therefore, they separate, after impact, a distance $=2e^2a\sin\frac{1}{4}\pi=e^2a\sqrt{2}$.

If $e=1$, they return to their starting points.

AVERAGE AND PROBABILITY.

190. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line is drawn at random across a regular $2n$-gon; what is the chance that it crosses parallel sides?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let AB, CD be two parallel sides. Through the center of the polygon O draw PQ to represent the direction of the random line.

Let $AO=r$, $\angle AOB=\pi/n=\beta$, $\angle AOP=\theta$.

For all possible positions PQ may be regarded as lying in one quadrant. The actual line parallel to PQ may hold any position within breadth of plane HF (H being the most remote vertex from QP, and F foot of perpendicular from H on QP) for all positions, and AE (A being the nearest vertex to QP and E foot of perpendicular from A to QP) for all favorable positions.

$\therefore$ The chance is $p=AE/HF$; $AE=r\sin\theta$, $HF=r\cos\theta$ for n even; $HF=r\cos(\frac{1}{2}\beta-\theta)$ for n odd. For n even,

$$p=\frac{2n}{\pi}\int_0^{\frac{1}{2}\beta}\tan\theta\,d\theta=\frac{2n}{\pi}\log(\sec\tfrac{1}{2}\beta). \quad \therefore p=\frac{2n}{\pi}\log\left(\sec\frac{\pi}{2n}\right).$$

For n odd, $p=\frac{2n}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin\theta}{\cos(\frac{1}{2}\beta-\theta)}d\theta=\frac{n}{\pi}(\beta\sin\tfrac{1}{2}\beta+2\cos\tfrac{1}{2}\beta\log\cos\tfrac{1}{2}\beta)$.

$$\therefore p=\frac{n}{\pi}\left[\frac{\pi}{n}\sin\frac{\pi}{2n}+2\cos\frac{\pi}{2n}\log\left(\cos\frac{\pi}{2n}\right)\right].$$

191. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random lines cut a given circle. What is the chance that they intersect within the circle?

Solution by HENRY HEATON, Belfield, N. D., and the PROPOSER.

Let x=the distance of one of the lines from the center of the circle, and θ=the angle between the lines. The length of the part of the first line lying within the circle is $2\sqrt{(a^2-x^2)}$. For given values of x and θ the chance of intersection is

$$\frac{2\sin\theta\sqrt{(a^2-x^2)}}{2a}=\frac{\sin\theta}{a}\sqrt{(a^2-x^2)},$$

and the required chance is

$$P=\int_0^a\int_0^{\frac{1}{2}\pi}\frac{\sin\theta}{a}\sqrt{(a^2-x^2)}\,dx\,d\theta\div\int_0^a\int_0^{\frac{1}{2}\pi}dx\,d\theta=\frac{2}{a\pi}\int_0^a\sqrt{(a^2-x^2)}\,dx=\tfrac{1}{2}.$$

Also solved by G. B. M. Zerr, who gets ⅓ for the result. His solution will be published in the next issue of the MONTHLY.

PROBLEMS FOR SOLUTION.

ALGEBRA.

293. Proposed by C. E. WHITE, Vanderbilt University, Nashville, Tenn.

Prove by mathematical induction that $\frac{(x-a)^{m-1}}{(m-1)!}f^{m-1}(a)+\frac{(x-a)^{m-2}}{(m-2)!}+\ldots.\frac{(x-a)^2}{2!}f''(a)+(x-a)f'(a)+f(a)$ will be the remainder when $f(x)$ is divided by $(x-a)^m$.

294. Proposed by O. L. CALLECOT, Gettysburg, S. Dak.

Find the limit of $\sum_{n=1}^{n=\infty}\frac{2(n^2+3n+3)}{n(n+1)(n+2)(n+3)}$.

GEOMETRY.

326. Proposed by L. E. NEWCOMB, Los Gatos, Calif.

The circle C of radius pR encloses the circles A_1B_1 of radii R and $(p-1)R$, respectively; the circle B_1 is tangent to $A_1B_1C_1$; the circle B_2 is tangent to AB_1C; the circle B_3 to AB_2C, ..., $B_n$to $AB_{n-1}C$. Find the radius of the circle B_n.

327. Proposed by J. C. CORBIN, Pine Bluff, Ark.

In triangle ABC, the triangle DEF is formed by joining the feet of the medians and four parallelograms are also formed, viz., $AEDF$, $BFED$, and $CEFD$. Let a, b, c, d, e, f represent the three medians of ABC, and the three sides of DEF. Then the sum of the squares of the six diagonals equals the sum of the squares of the twelve sides of the parallelograms, which are equal in sets of four. That is, $a^2+b^2+c^2+d^2+e^2+f^2=4(d^2+e^2+f^2)$, or $a^2+b^2+c^2=3(d^2+e^2+f^2)=3/4(AB^2+BC^2+CA^2)$.

328. Proposed by CHARLES GILPIN, JR., Philadelphia, Pa.

A sphere with the radius R is divided into two segments by a plane passed through it half way between the center and circumference. The smaller segment is divided into two parts by a plane passed through it at right angles to the base and cutting it half way between its center and circumference. Required the contents of the two parts of the segment.

CALCULUS.

250. Proposed by V. M. SPUNAR, East Pittsburg, Pa.

Differentiate $(\log^n x)$.

251. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Find in terms of x the functions c_1x and c_2x defined, respectively, by the relations

(a) $x\log(c_1x)=c_1x\log x$,

(b) $x\log x=c_2x\log(c_2x)$.

MECHANICS.

210. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A rigid triangle is formed of three weightless, smoothly jointed, rigid rods BC, CA, AB. At their mid points D, E, F, respectively, are small, smooth rings, through which passes an endless, stretched, elastic string, forming the triangle DEF. Find by graphical construction the reaction at the joints.

211. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A smooth elliptic wire, axis vertical, has a small ring sliding on it, connected by elastic strings with each focus. Either string is just unstretched when the ring is nearest the corresponding focus. The modulus of elasticity is W/n, where W oz. is the weight of the ring. Find the distance of the ring from the upper focus in the different positions of equilibrium, and in each case discuss the nature of the equilibrium.

MISCELLANEOUS.

175. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

If x and z are connected by the relation $z=zf(x)+x\phi(z)$, find the value of $f(z)$ in the form of a power series in x with constant coefficients. In particular, give such a value of z when $z=z\sin x+x\cos z$.

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